The Path of an Electron in Combined Radial Magnetic and Electric Fields.

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1. In the course of an investigation with a focus tube, in which the anti-cathode was a magnetic pole, some interesting observations were made with regard to the distribution of the fluorescence on the walls of the tube as the magnetic field varied. In order to explain the changes observed, the path of an electron in a radial field was investigated mathematically, and was found to lie on a right circular cone whose vertex coincides with the magnetic pole. If the surface of the cone is developed into a plane the trace of the path is a conic section, with the vertex as focus. This result continues to hold when an electric field, with its lines of force radiating from the same point, is superposed on the magnetic field. Although the actual distribution of electric intensity in a highly exhausted tube* must differ widely from that here considered, the investigation sufficiently explains the facts observed.

Theoretical.

2. The object is to find the path of a particle of mass m carrying a charge e in combined magnetic and electric fields, when the lines of force are radial and are such as might be due to a single pole of strength μ coincident with an electric charge k.

The equations of motion of the particle are

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= e\left(\frac{\mu y}{r^3}\frac{dz}{dt}-\frac{\mu z}{r^3}\frac{dy}{dt}+\frac{kx}{r^3}\right) \\ m\frac{d^2y}{dt^2} &= e\left(\frac{\mu z}{r^3}\frac{dx}{dt}-\frac{\mu x}{r^3}\frac{dz}{dt}+\frac{ky}{r^3}\right) \\ m\frac{d^2z}{dt^2} &= e\left(\frac{\mu x}{r^3}\frac{dy}{dt}-\frac{\mu y}{r^3}\frac{dx}{dt}+\frac{kz}{r^3}\right) \end{aligned}\right\} \qquad (1)$$

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* F. W. Aston, 'Roy. Soc. Proc.,' January, 1911, vol. 84, pp. 526—535.

If we multiply the second equation by z, the third equation by y, and subtract, we obtain on integration

$$\left.\begin{aligned} & z\frac{dy}{dt}-y\frac{dz}{dt}=\frac{\mu e}{m}\left(\frac{x}{r}-\mathrm{A}\right). \\ \text{Similarly}\qquad & x\frac{dz}{dt}-z\frac{dx}{dt}=\frac{\mu e}{m}\left(\frac{y}{r}-\mathrm{B}\right) \\ \text{and}\qquad & y\frac{dx}{dt}-x\frac{dy}{dt}=\frac{\mu e}{m}\left(\frac{z}{r}-\mathrm{C}\right). \end{aligned}\right\} \qquad (2)$$

Multiply these equations by x, y, z, respectively and add.

Then
$$\mathrm{A}x+\mathrm{B}y+\mathrm{C}z=r, \qquad (3)$$
showing that the particle always lies on a right circular cone whose vertex is the origin.

3. It is easy to show from the equations of motion that the velocity V at a distance r from the origin is given by

$$\mathrm{V}^2-\mathrm{V}_0{}^2=\frac{2ke}{m}\left(\frac{1}{r_0}-\frac{1}{r}\right), \qquad (4)$$

where V_0, r_0, are the corresponding values for the point of projection. This is, in fact, the equation of energy.

If the particle is moving with velocity V in a direction making an angle β with a generating line OP of the cone, its acceleration along the normal is equal to $\mathrm{V}^2\sin^2\beta/\mathrm{PG}$, while the force component in that direction is due to the action of magnetic field μ/r^2.

Hence
$$\frac{m\mathrm{V}^2\sin^2\beta}{\mathrm{PG}}=\frac{\mu}{r^2}e\mathrm{V}\sin\beta,$$

so that
$$\mathrm{V}r\sin\beta=\frac{\mu e}{m}\tan\alpha \qquad (5)$$
$$= \text{a constant, say } \mathrm{V}_0r_0\sin\beta_0.$$

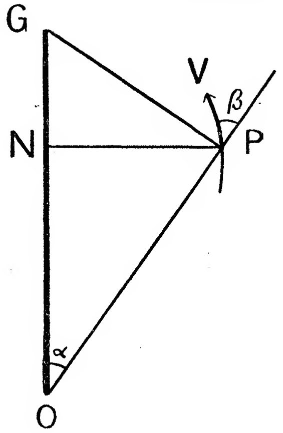

Fig. 1.

Since $r\sin\beta = p$, the length of the perpendicular from the origin on the tangent to the orbit, this result may be written

$$\mathrm{V}p = \frac{\mu e}{m}\tan\alpha = \mathrm{V}_0 p_0. \tag{6}$$

Thus, if the surface of the cone is developed into a plane, the velocity is that due to a central force ke/r^2, and $\mathrm{V}p$ is constant, so that the curve is a central orbit; it is therefore a conic section with the origin as focus.

If k and e have the same sign the force due to the electric field is one of repulsion, and the conic is an hyperbola. If k and e have opposite signs the force is an attraction and the curve is an hyperbola, an ellipse, or a parabola, according to the circumstances of projection.

The semi-axes of the conic are given by the relations

$$\pm a = \frac{ke}{m}\Big/\left(\mathrm{V}_0^2 + \frac{2ke}{mr_0}\right), \quad \pm b^2 = \mathrm{V}_0^2 p_0^2\Big/\left(\mathrm{V}_0^2 + \frac{2ke}{mr_0}\right).$$

If $\mathrm{V}_0^2 = 2ke/mr_0$, the conic is a parabola whose latus rectum is $4p_0^2/r_0$.

4. Two particular cases merit special attention.

Case I. Motion of an Electron in a Radial Electric Field.—The surface of the cone becomes a plane.

Case II. Motion of an Electron in a Radial Magnetic Field.—The solution has been given by Poincaré.*

Putting $k = 0$ in the previous results we find that the velocity of the particle is constant, and that consequently $r\sin\beta$ or p is a constant. This implies that when the surface of the cone is developed into a plane the trace of the path is a straight line.

The number of times that the path on the cone overlaps itself depends on the value of the vertical angle ($2\pi/\sin\alpha$) of one sheet of the developed cone. If this angle is greater than π there is no point of overlapping, if its value be between π and $\frac{1}{2}\pi$ there is one such point, and if its value be between $\pi/2(n-1)$ and $\pi/2n$ there will be n such points, n being a positive integer.

5. *Numerical Estimates.*—It is instructive to consider the magnitude of the quantities determining the path of an electron ($e/m = 1{\cdot}7\times10^7$ E.M. units) projected with a velocity of 10^8 or 10^9 cm. per sec.

The value of the semi-angle α of the cone in certain cases is given in Table I:—

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* 'Comptes Rendus,' 1896, vol. 123, pp. 530—533. [Sir Joseph Larmor has pointed out that the mathematically equivalent problem of finding the form assumed by a flexible string carrying an electric current in a radial magnetic field has been solved analytically by Darboux. The solution can be obtained directly by an application of the Principle of Least Action. See 'Proc. Lond. Math. Soc.,' 1884, vol. 15, pp. 158—170.]

Table I.

	$V_0 = 10^8$ cm./sec.			$V_0 = 10^9$ cm./sec.		
	$\mu = 50$.	$\mu = 500$.	$\mu = 5000$.	$\mu = 50$.	$\mu = 500$.	$\mu = 5000$.
$p_0 = 1$ cm.	6° 43′	0° 40′	0° 4′	49° 38′	6° 43′	0° 40′
$p_0 = 2$ cm.	13° 45′	1° 21′	0° 8′	66° 58′	13° 45′	1° 21′

The strength of the radial electric field does not affect the angle of the cone, but determines the character of the developed path. Under the conditions met with in an ordinary focus tube the curve is an hyperbola differing but little from a straight line. Take, for example, the case in which $V_0 = 10^9$ cm./sec., $p_0 = 1$ cm., $r_0 = 6$ cm., and the electron is attracted towards the origin. In order to obtain a parabolic curve on the developed cone we should have to make $2ke/mr_0 = V_0^2$ by giving k a value of approximately 2×10^{11} *electromagnetic* units. By giving smaller values to k the curve becomes hyperbolic, and as k is diminished the semi-major axis a becomes small, the semi-minor axis b becomes nearly equal to p_0, and the distance from the focus (the vertex of the cone) to the vertex of the hyperbola approaches the same limit, p_0. In order to obtain an electric field whose strength at a few centimetres from the origin is measured in hundreds of volts per centimetre we have to assign extremely large values to k, but even then the developed path will not in general differ greatly from a straight line.

Experimental.

6. My attention was first drawn to the behaviour of the cathode stream in a radial magnetic field when using a focus tube of the form shown in fig. 2. The iron core of an electromagnet passed up through a barometer tube into the centre of the bulb, so that a strong magnetic pole could be produced at A. The form of the cathode C happened to be such that, when the bulb was highly exhausted, a ring of bright green fluorescence was seen on the glass at BB, encircling the iron bar. When the magnetic field was produced, this bright ring moved upwards till it reached a position such as DD. On gradually increasing the strength of the field, the bright ring rose higher to the position shown at EE. The lower part of the bulb was now practically free from luminosity. As the fluorescence on the wall of the bulb ascended, a violet cone of light made its appearance in the interior, the vertex of the cone being at the magnetic pole A. The boundary of the cone

for a strong magnetic field is shown by the dotted lines AE. This luminous cone was visible also at higher pressures when there was no fluorescence on the glass walls.

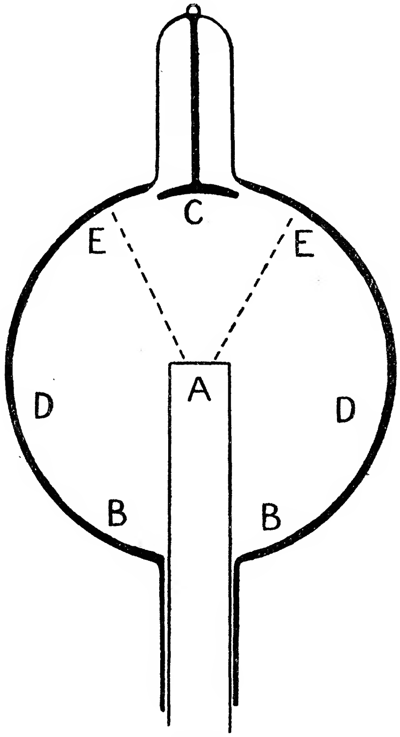

Fig. 2.

The explanation of these results by means of the theory already given is simple. The boundary of the violet cone seen in the bulb is the envelope of all the cones on which the spiral paths of the electrons lie. As the strength of the magnetic field is increased the vertical angles of these cones diminish, and consequently the angle of the enveloping cone also diminishes. The ring of fluorescence marks the curve in which the enveloping cone meets the glass of the bulb. As the field is increased the fluorescent ring gradually ascends the walls of the tube.

7. Experiments illustrating the behaviour of the cathode stream in a magnetic field due to a single pole have been carried out by Birkeland* and discussed by Poincaré.†

In order to illustrate further the case of Poincaré, in which no electric field acts on the particle, and the path is a geodesic line on a right circular cone, I have experimented with vacuum tubes in which a fine pencil of cathode rays could be produced. A Wehnelt cathode was formed by melting a small speck of sealing-wax on a fine strip of platinum foil. When the foil was heated to a dull red heat by a current of 5 or 6 ampères, a potential difference of 100 or 200 volts was sufficient to maintain the discharge between the cathode and an aluminium anode placed in a side tube. The slow-moving stream of cathode particles was then subjected to the influence of one pole of a long electromagnet, and the path of the stream observed and photographed. In one tube the magnetic pole was situated in the centre of the spherical bulb through which the stream of electrons passed, in a second tube the pole was placed at any point outside the glass wall of the bulb. The strength of the pole could be increased up to about 1000 C.G.S. units. The spiral paths observed were found to lie on a cone, with its vertex at the magnetic pole.

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* 'Archives des Sciences' (Geneva), 1896, Ser. 4, vol. 1, pp. 407—512.

† *Loc. cit.*

In conclusion, I desire to express my thanks to Profs. C. G. Barkla and J. B. Dale for their interest in this work, and to the Government Grant Committee of the Royal Society for some of the apparatus employed.

On the Dynamical Nature of the Molecular Systems which emit Spectra of the Banded Type.

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§ 1. *Summary of Paper.*—It is now widely believed, for the reasons recounted in § 2 below, that when the spectrum emitted by a luminous body is of the banded type, the small vibrators which give rise to the radiation are the molecules of the substance, as distinguished from atoms or ions. This result is applied in § 3, which constitutes the main body of the present paper, in order to suggest a dynamical system which is formed of two members in the same way as a diatomic molecule may be supposed to be formed of two atoms, and which has free periods of vibration related to each other by the same formula as holds in the case of banded spectra. This formula presents a certain peculiarity, in that the frequency of vibration occurs in it linearly; whereas in the equation for determining the free periods of dynamical systems in general, the frequency enters by its square. It is